A Simple Optimal Contention Resolution Scheme for Uniform Matroids

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Aussois 2024

Fair Contention Resolution [Feige, Vondrák (2006)]

- $N = \{1, \ldots, n\}$ players and one item
- Each $i \in N$ independently requests the item with probability $x_i \in [0, 1]$
- Assume that $x_1 + \cdots + x_n \leq 1$

Goal: assign the item to at most one player and maximize $\rho \in [0, 1]$ s.t.

$$\mathbb{P}\Big[\ i \text{ gets item} \ \Big| \ i \text{ requests item} \ \Big] \geq \rho \quad \forall i \in N$$
$$\iff \mathbb{P}\Big[\ i \text{ gets item} \ \Big] \geq \rho\, x_i \quad \forall i \in N$$

Applications: Leads to approximation algorithms for combinatorial allocation problems (Submodular Welfare, Generalized Assignment, . . .)

Fair Contention Resolution

Let $A \subseteq N$ be the random set of players requesting the item

Optimal algorithm [Feige, Vondrák (2006)]

- if $A = \emptyset$, do not allocate the item
- if $A = \{k\}$, allocate to player i
- if $|A| > 1$, allocate to each $i \in A$ with probability

$$r_A(i) := \frac{1}{x(N)} \left(\frac{x(A \setminus i)}{|A| - 1} + \frac{x(N \setminus A)}{|A|} \right)$$

Theorem. This algorithm achieves an optimal balancedness of

$$\rho = 1 - \left(1 - \frac{1}{n}\right)^n$$

Asymptotically converges from above to $1 - 1/e \approx 0.63$

Contention Resolution Schemes

- Ground set $N = \{1, \ldots, n\}$
- Feasible sets $\mathcal{I} \subseteq 2^N$ which are downward-closed (if $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$)
- Relaxation polytope $P_{\mathcal{I}} \subseteq [0, 1]^N$
- Fractional point $x \in P_{\mathcal{I}}$

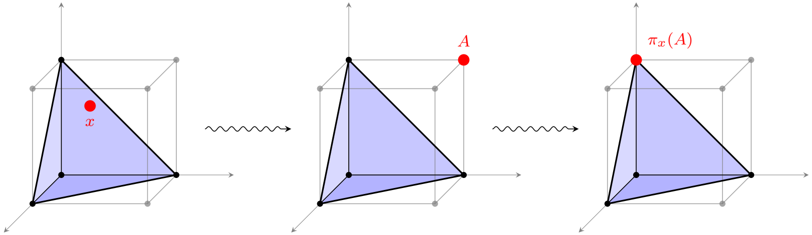
CR schemes [Chekuri, Vondrák, Zenklusen (2014)]

1. Each $i \in N$ independently rounds to $\{0, 1\}$ with probability x_i, giving a random set of active elements $A \subseteq N$
2. **Goal:** remove elements from A to get a feasible set $\pi_x(A) \in \mathcal{I}$ s.t.

$$\mathbb{P}\Big[\, i \in \pi_x(A) \;\Big|\; i \in A \,\Big] \geq \rho \quad \forall i \in N$$

Contention Resolution Schemes

$$P_{\mathcal{I}} = \{x \in [0,1]^3, x_1 + x_2 + x_3 \leq 1\}:$$



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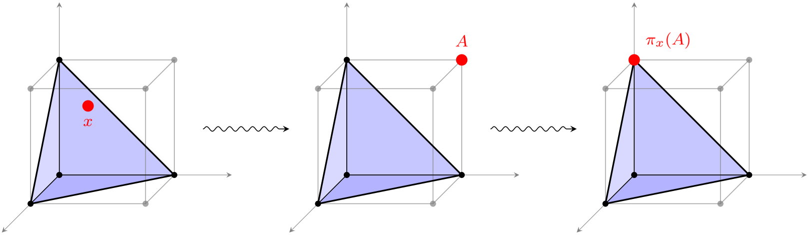
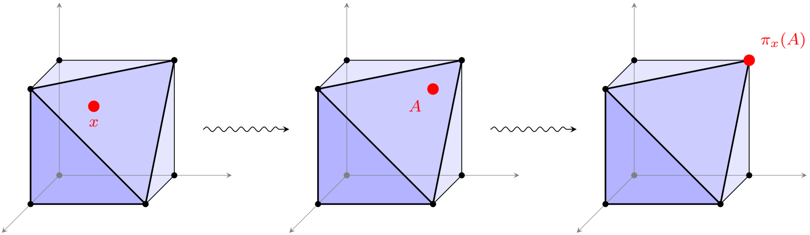


$$P_{\mathcal{I}} = \{x \in [0,1]^3, x_1 + x_2 + x_3 \leq 2\}:$$



Known Results

We are interested in uniform matroids $U(k, n)$ with

$$\mathcal{I} = \{S \subseteq N, |S| \leq k\} \qquad P_{\mathcal{I}} = \{x \in [0,1]^N, x(N) \leq k\}$$

Known Results

1. $1 - (1 - 1/n)^n$ balanced scheme for $U(1, n)$ [Feige, Vondrák (2006)]
2. $1 - 1/e$ balanced scheme for general matroids [Chekuri et. al. (2014)]
3. $1 - e^{-k}k^k/k!$ correlation gap for $U(k, n)$ [Yan (2010)]

- 1. is optimal for any $n \in \mathbb{N}$, converges to $1 - 1/e$ as $n \to \infty$
- 2. and 3. are asymptotically optimal as $n \to \infty$.

Our Results

Simple optimal scheme for $U(k, n)$

Input: $x \in P_{\mathcal{I}}, A \subseteq N$

- If $|A| \leq k$, then $\pi_x(A) = A$
- If $|A| > k$, then sample from $\{B \subset A, |B| = k\}$ with probability

$$q_A(B) := \frac{1}{\binom{|A|}{k}}\Big(1 + \bar{x}(A \setminus B) - \bar{x}(B)\Big)$$

Theorem

1. This algorithm is $c(k, n)$-balanced with

$$c(k, n) := 1 - \binom{n}{k} \left(1 - \frac{k}{n}\right)^{n+1-k} \left(\frac{k}{n}\right)^{k}$$

2. Optimal for any values of k and n.

Where does $c(k,n)$ come from?

$$h(x) := \sum_{A \subseteq N, |A|=k} \Big(1 - \bar{x}(A)\Big) \prod_{j \in A} x_j \prod_{j \notin A} (1 - x_j)$$

Evaluating $h(x)$ at $(k/n, \ldots, k/n)$ gives

$$\sum_{A \subseteq N, |A|=k} \left(1 - \frac{k}{n}\right) \left(\frac{k}{n}\right)^k \left(1 - \frac{k}{n}\right)^{n-k}$$

$$= \binom{n}{k} \left(1 - \frac{k}{n}\right)^{n+1-k} \left(\frac{k}{n}\right)^k = 1 - c(k,n)$$

Note: $c(k,n)$ converges from above to $1 - e^{-k} k^k / k!$ as $n \to \infty$.

Comparison to previous scheme for $U(1, n)$

Gives another optimal scheme for $U(1, n)$ with $\rho = 1 - (1 - 1/n)^n$

Previous scheme

If $|A| > 1$, allocate to each $i \in A$ with probability

$$r_A(i) = \frac{1}{x(N)} \left(\frac{x(A \setminus i)}{|A| - 1} + \frac{x(N \setminus A)}{|A|} \right)$$

This scheme

If $|A| > 1$, allocate to each $i \in A$ with probability

$$q_A(i) = \frac{1}{|A|} \left(1 + \bar{x}(A \setminus i) - x_i \right)$$

- Linear with respect to x
- Generalizes to any k

Efficient implementation for $U(k, n)$

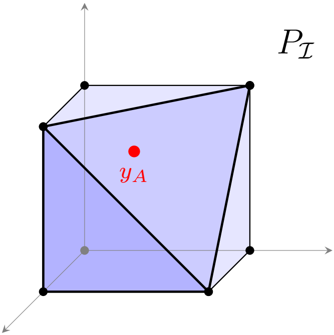
For $U(k, n)$, it suffices to know the marginals of the distribution q_A.

Lemma

For every $|A| > k$ and every $i \in A$:

$$y_A(i) := \mathbb{P}\Big[i \in \pi_x(A)\Big] = \frac{1}{|A|}\Big(k + \bar{x}(A \setminus i) - x_i\Big)$$

- $y_A \in P_{\mathcal{I}}$, since q_A is a distribution over $\mathcal{I}$
- Find a convex combination of at most $n + 1$ vertices
- Sample a vertex according to that distribution



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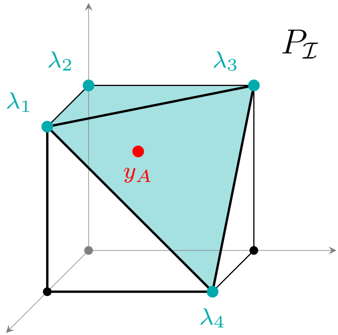

Proof idea

Fix $i \in N$. We need to show $\forall x \in P_{\mathcal{I}}$:

$$G(x) := \mathbb{P}\Big[i \notin \pi_x(A) \mid i \in A \Big] \leq 1 - c(k,n).$$

Use the law of total probability with q_A to get a multivariable polynomial function of x.

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Use the law of total probability with q_A to get a multivariable polynomial function of x.

Goal:

$$\max_{x \in P_{\mathcal{I}}} G(x) = 1 - c(k,n)$$

1. Use linearity of the scheme and first maximize $G(x)$ over x_i
2. Maximize the obtained symmetric function $h(x_{-i})$ in the variables

$$x_{-i} := \Big\{x_j,\ j \in N \setminus \{i\}\Big\}$$

$$P_{\mathcal{I}} = \{x \in [0,1]^N, x_1 + \cdots + x_n \leq k\}$$

$G(x)$ is a linear function of x_i if considering the other variables fixed. Moreover, the linear coefficient in front of x_i is positive.

$$\to x_i = k - x(N \setminus \{i\})$$

Lemma

$G(x) \leq h(x_{-i})$ for every $x \in P_{\mathcal{I}}$ with equality at $x_i = k - x(N \setminus \{i\})$

$$h(x_{-i}) := \sum_{A \subseteq N \setminus \{i\}, |A| = k} \Big(1 - \bar{x}(A)\Big) \prod_{j \in A} x_j \prod_{j \notin A} (1 - x_j)$$

$$h(x_{-i}) = \sum_{A \subseteq N \setminus \{i\}, |A|=k} \Big(1 - \bar{x}(A)\Big) \prod_{j \in A} x_j \prod_{j \notin A} (1 - x_j)$$

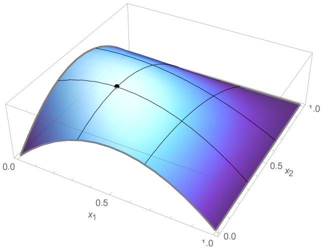

Figure:
$n = 3, k = 1,$ maximum at $(1/3, 1/3)$

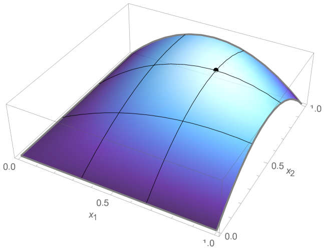

Figure:
$n = 3, k = 2,$ maximum at $(2/3, 2/3)$

Lemma

$h(x_{-i})$ has a unique extremum (maximum) at $(k/n, \ldots, k/n)$

Evaluating $h(x_{-i})$ at $x_{-i} = (k/n, \ldots, k/n)$ still gives

$$h(x_{-i}) = \sum_{A \subseteq N \setminus \{i\}, |A|=k} \Big(1 - \bar{x}(A)\Big) \prod_{j \in A} x_j \prod_{j \notin A} (1 - x_j)$$
$$= \binom{n-1}{k} \left(1 - \frac{k}{n}\right)^{n-k} \left(\frac{k}{n}\right)^{k}$$
$$= \binom{n}{k} \left(1 - \frac{k}{n}\right)^{n+1-k} \left(\frac{k}{n}\right)^{k} = 1 - c(k, n)$$

Theorem (Balancedness)

The *CR* scheme for $U(k, n)$ has balancedness $\rho = c(k, n)$.

□

Hardness

Theorem (Hardness)

No CR scheme for $U(k, n)$ can have balancedness $\rho > c(k, n)$

- Fix $x_i = k/n$ for every $i \in N$
- Let π be a ρ -balanced CR scheme returning $\pi_x(A)$

$$\mathbb{E}\Big[\text{rank}(A)\Big] \geq \mathbb{E}\Big[|\pi_x(A)|\Big] = \sum_{i \in N} \mathbb{P}\Big[i \in \pi_x(A)\Big] \geq \sum_{i \in N} \rho\, x_i = \rho\, k$$

Lemma

At the point $x = (k/n, \ldots, k/n)$:

$$\mathbb{E}\Big[\text{rank}(A)\Big] = k\Big(1 - h(x)\Big)$$

$$\implies \rho \leq 1 - h(x) = c(k, n) \quad \square$$

Conclusion

Conclusion

- Simple optimal CR scheme for uniform matroids
- Different optimal probability distribution for $U(1, n)$
- Generalizes to $U(k, n)$ for every k, n
- Upper and lower bounds depending on $n \in \mathbb{N}$

Future work ideas

- Other simple CR schemes for different constraint families
- Different (simpler) proof of the optimality of this algorithm
- Try to adapt it in an online or random order model

Thanks!